THE CHINESE UNIVERSITY OF HONG KONG
MATH4010 Suggested solutions to homework 5

If you find any mistakes or typos, please report them to ypyang@math.cuhk.edu.hk

7.17. Let X be an inner product space and E a nonempty subset of X. Show that

$$E^\perp = \bigcap_{x\in E} \{x\}^\perp.$$

Proof. For any $z \in E^\perp$, we have $\langle z, x\rangle = 0, \forall x \in E$, which implies that $z \in \{x\}^\perp$. Therefore,

$$E^\perp \subset \bigcap_{x\in E} \{x\}^\perp.$$

On the other hand, for any $y \in \bigcap_{x\in E} \{x\}^\perp$, we have $y \in \{x\}^\perp, \forall x \in E$. It follows that $\langle y, x\rangle = 0, \forall x \in E$ and consequently $y \in E^\perp$. As a result, $\bigcap_{x\in E} \{x\}^\perp \subset E^\perp$.

7.18. (a) Prove that for every two subspaces X_1 and X_2 of a Hilbert space,

$$(X_1 + X_2)^\perp = X_1^\perp \cap X_2^\perp.$$

(b) Prove that for every two closed subspaces X_1 and X_2 of a Hilbert space,

$$(X_1 \cap X_2)^\perp = \overline{X_1^\perp + X_2^\perp}.$$

Proof. (a) First note that if $W \subset V$ then $V^\perp \subset W^\perp$.
Now $X_1, X_2 \subset X_1 + X_2$ and so we have

$$(X_1 + X_2)^\perp \subset X_1^\perp,\ \ (X_1 + X_2)^\perp \subset X_2^\perp \Longrightarrow (X_1 + X_2)^\perp \subset X_1^\perp \cap X_2^\perp.$$

Now we need to show the other way of inclusion. Let $x \in X_1^\perp \cap X_2^\perp$ and z be an arbitrary vector in $X_1 + X_2$. Then we can write $z = x_1 + x_2$, $x_1 \in X_1, x_2 \in X_2$ and $\langle x, z\rangle = \langle x, x_1\rangle + \langle x, x_2\rangle = 0$. This follows because x is in both $X_1^\perp$ and $X_2^\perp$. Therefore, $x \in (X_1 + X_2)^\perp$ and

$$X_1^\perp \cap X_2^\perp \subset (X_1 + X_2)^\perp.$$

And we are done.
(b) Since X_1 and X_2 are closed, from (a) we have

$$(\overline{X_1^\perp + X_2^\perp})^\perp = (X_1^\perp + X_2^\perp)^\perp = X_1^{\perp\perp} \cap X_2^{\perp\perp} = X_1 \cap X_2.$$

Therefore,

$$(X_1 \cap X_2)^\perp = (\overline{X_1^\perp + X_2^\perp})^{\perp\perp} = \overline{X_1^\perp + X_2^\perp}$$

where the last identity is from the closedness of $\overline{X_1^\perp + X_2^\perp}$.

7.26. Let (e_k) be an orthonormal sequence in an inner product space X. Prove that

$$\sum_{k=1}^{\infty} | \langle x, e_k \rangle \langle y, e_k \rangle | \leq \|x\| \|y\|,$$

for all $x, y \in X$.

Proof. From the Cauchy-Schwarz inequality and Bessel's inequality, we have

$$\sum_{k=1}^{\infty} | \langle x, e_k \rangle \langle y, e_k \rangle | \leq \left(\sum_{k=1}^{\infty} | \langle x, e_k \rangle |^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} | \langle y, e_k \rangle |^2 \right)^{\frac{1}{2}} \leq \|x\| \|y\|.$$